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A SOLUTION OF EINSTEIN'S FIELD EQUATIONS FOR THE THIRD CLASS IN X_4

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Abstract. The main goal in the present paper is to obtain a solution of Einstein's unified field equations for the third class in X_4.

1. Introduction

Einstein([1]) proposed a new unified field theory that would include both gravitation and electromagnetism. Hlavatý([5]) gave the mathematical foundation of the Einstein's unified field theory in a 4-dimensional generalized Riemannian space X_4 (i.e., space-time) for the first time. Since then this theory had been generalized in a generalized Riemannian manifold X_n, the so-called *Einstein's n-dimensional unified field theory*, and many consequences of this theory has been obtained by a number of mathematicians. However, it has been unable yet to represent a general n-dimensional Einstein's connection in a surveyable tensorial form, probably due to the complexity of the higher dimensions. The purpose of the present paper is to obtain a necessary and sufficient condition for a connection with a new torsion tensor be an Einstein's connection in X_n. In the next, we obtain a solution of Einstein's field equations for the third class in X_4. The obtained results and discussions in the present paper will be useful for the 4-dimensional considerations of the unified field theory.

2. Preliminary

Let X_n be an n-dimensional generalized Riemannian manifold covered by a system of real coordinate neighborhoods $\{U; x^\nu\}$, where, here and

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in the sequel, Greek indices run over the range $\{1, 2, \cdots, n\}$ and follow the summation convention. The algebraic structure on X_n is imposed by a basic real non-symmetric tensor $g_{\lambda\mu}$, which may be split into its symmetric part $h_{\lambda\mu}$ and skew-symmetric part $k_{\lambda\mu}$:

$$g_{\lambda\mu} = h_{\lambda\mu} + k_{\lambda\mu}, \tag{2.1}$$

where we assume that

$$(a)\ G = det((g_{\lambda\mu})) \neq 0, \quad (b)\ H = det((h_{\lambda\mu})) \neq 0. \tag{2.2}$$

Since $det((h_{\lambda\mu})) \neq 0$, we may define a unique tensor $h^{\lambda\nu}(= h^{\nu\lambda})$ by

$$h_{\lambda\mu}h^{\lambda\nu} = \delta^{\nu}_{\mu}. \tag{2.3}$$

We use the tensors $h^{\lambda\nu}$ and $h_{\lambda\mu}$ as tensors for raising and/or lowering indices for all tensors defined on X_n in the usual manner. Then we may define new tensors by

$$(a)\ k^{\alpha}{}_{\mu} = k_{\lambda\mu}h^{\lambda\alpha}, \quad (b)\ k_{\lambda}{}^{\alpha} = k_{\lambda\mu}h^{\mu\alpha}, \quad (c)\ k^{\alpha\beta} = k_{\lambda\mu}h^{\lambda\alpha}h^{\mu\beta}. \tag{2.4}$$

The manifold X_n is assumed to be connected by a general real connection $\Gamma^{\nu}_{\lambda\mu}$ which may also be split into its symmetric part $\Lambda^{\nu}_{\lambda\mu}$ and skew-symmetric part $S_{\lambda\mu}{}^{\nu}$, called the *torsion tensor* of $\Gamma^{\nu}_{\lambda\mu}$:

$$\begin{aligned} &(a)\ \Lambda^{\nu}_{\lambda\mu} = \Gamma^{\nu}_{(\lambda\mu)} = \frac{1}{2}(\Gamma^{\nu}_{\lambda\mu} + \Gamma^{\nu}_{\mu\lambda}), \\ &(b)\ S_{\lambda\mu}{}^{\nu} = \Gamma^{\nu}_{[\lambda\mu]} = \frac{1}{2}(\Gamma^{\nu}_{\lambda\mu} - \Gamma^{\nu}_{\mu\lambda}). \end{aligned} \tag{2.5}$$

The *Einstein's n-dimensional unified field theory in X_n* is governed by the following set of equations :

$$\partial_{\omega}g_{\lambda\mu} - g_{\alpha\mu}\Gamma^{\alpha}_{\lambda\omega} - g_{\lambda\alpha}\Gamma^{\alpha}_{\omega\mu} = 0 \qquad (\partial_{\nu} = \frac{\partial}{\partial x^{\nu}}), \tag{2.6}$$

and

$$(a)\ S_{\lambda} = S_{\lambda\alpha}{}^{\alpha} = 0, \quad (b)\ R_{[\lambda\mu]} = \partial_{[\lambda}P_{\mu]}, \quad (c)\ R_{(\lambda\mu)} = 0, \tag{2.7}$$

where P_{μ} is an arbitrary vector, called the *Einstein's vector*, and $R_{\lambda\mu}$ is the contracted curvature tensor $R^{\alpha}_{\lambda\mu\alpha}$ of the curvature tensor $R^{\omega}_{\lambda\mu\nu}$:

$$R^{\omega}_{\lambda\mu\nu} = \partial_{\mu}\Gamma^{\omega}_{\lambda\nu} - \partial_{\nu}\Gamma^{\omega}_{\lambda\mu} + \Gamma^{\alpha}_{\lambda\nu}\Gamma^{\omega}_{\alpha\mu} - \Gamma^{\alpha}_{\lambda\mu}\Gamma^{\omega}_{\alpha\nu}. \tag{2.8}$$

The equation (2.6) is called the *Einstein's equation*, and the solution $\Gamma^{\nu}_{\lambda\mu}$ of the Einstein's equation is called an *Einstein's connection*. And the vector S_{λ}, defined by (2.7)(a), is called the *torsion vector*.

3. An Einstein's connection in X_n

The following theorem was proved by Hlavatý([5]).

THEOREM 3.1. *In X_n, if the Einstein's equation (2.6) admits a solution $\Gamma^{\nu}_{\lambda\mu}$, then this solution must be of the form*

$$\Gamma^{\nu}_{\lambda\mu} = \{{}_{\lambda}{}^{\nu}{}_{\mu}\} + 2h^{\nu\alpha} S_{\alpha(\lambda}{}^{\beta} k_{\mu)\beta} + S_{\lambda\mu}{}^{\nu}, \tag{3.1}$$

where $\{{}_{\lambda}{}^{\nu}{}_{\mu}\}$ are the Christoffel symbols defined by $h_{\lambda\mu}$.

REMARK 3.2. In virtue of Theorem 3.1, the equation (3.1) reduces the investigation of $\Gamma^{\nu}_{\lambda\mu}$ to the study of its torsion tensor $S_{\lambda\mu}{}^{\nu}$. Hence in order to know an Einstein's connection $\Gamma^{\nu}_{\lambda\mu}$, it is necessary and sufficient to know its torsion tensor $S_{\lambda\mu}{}^{\nu}$. For this, we introduce a new torsion tensor $S_{\lambda\mu}{}^{\nu}$ given by

$$S_{\lambda\mu}{}^{\nu} = 2\delta^{\nu}_{[\lambda} k_{\mu]\alpha} Y^{\alpha} + k_{\lambda\mu} Y^{\nu}, \tag{3.2}$$

for some nonzero vector Y_{λ}.

THEOREM 3.3. *In X_n, if the connection (3.1) is a connection such that its torsion tensor is of the form (3.2) for some nonzero vector Y_{λ}, then the connection is given by*

$$\Gamma^{\nu}_{\lambda\mu} = \{{}_{\lambda}{}^{\nu}{}_{\mu}\} + 2\delta^{\nu}_{[\lambda} k_{\mu]\alpha} Y^{\alpha} + k_{\lambda\mu} Y^{\nu}. \tag{3.3}$$

Proof. Since the torsion tensor of the connection (3.1) is of the form (3.2), we obtain

$$2h^{\nu\alpha} S_{\alpha(\lambda}{}^{\beta} k_{\mu)\beta} = 0 \tag{3.4}$$

by a straightforward computation. Substituting (3.2) and (3.4) into (3.1), we obtain (3.3). □

THEOREM 3.4. *In X_n, the connection (3.3) is an Einstein's connection if and only if the vector Y_{λ} defining (3.3) satisfies the following condition*

$$\nabla_{\nu} k_{\lambda\mu} = 2h_{\nu[\lambda} k_{\mu]\alpha} Y^{\alpha} - 2k_{\nu[\lambda} Y_{\mu]}, \tag{3.5}$$

where ∇_{ω} is the symbolic vector of the covariant derivative with respect to $\{{}_{\lambda}{}^{\nu}{}_{\mu}\}$.

Proof. The connection (3.3) is an Einstein's connection if and only if the connection (3.3) satisfies the Einstein's equation (2.6). Substituting (2.1) and (3.3) into (2.6), and making use of $\nabla_{\nu} h_{\lambda\mu} = 0$, we obtain

$$\nabla_{\nu} k_{\lambda\mu} - 2h_{\nu[\lambda} k_{\mu]\alpha} Y^{\alpha} + 2k_{\nu[\lambda} Y_{\mu]} = 0 \tag{3.6}$$

by a straightforward computation. Hence the connection (3.3) is an Einstein's connection if and only if the vector Y_λ defining (3.3) satisfies the condition (3.5). □

THEOREM 3.5. *In X_n, if the connection (3.3) is an Einstein's connection, then its torsion vector satisfies the following relation :*

$$S_\lambda = \nabla_\alpha {k_\lambda}^\alpha \tag{3.7}$$

Proof. Contracting for (3.2) for μ and ν, we obtain

$$S_\lambda = {S_{\lambda\alpha}}^\alpha = (2-n) k_{\lambda\alpha} Y^\alpha. \tag{3.8}$$

Next, multiplying $h^{\mu\alpha}$ on both sides of (3.5), and contracting for ν and α, we obtain

$$\nabla_\alpha {k_\lambda}^\alpha = (2-n) k_{\lambda\alpha} Y^\alpha. \tag{3.9}$$

The results (3.8) and (3.9) imply the relation (3.7). □

4. A solution of field equations for the third class in X_4

In this section we shall display a solution for the third class of (2.6) and (2.7) in X_4. Assume $h_{\lambda\mu}$ to be of the form

$$((h_{\lambda\mu})) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{4.1}$$

Define two vectors by

$$(a)\ A_\lambda : (0, 0, 1, -1), \qquad (b)\ B_\lambda : (\phi, \psi, 0, 0), \tag{4.2}$$

where $\phi = \phi(x_1, x_2, x_3, x_4)$ and $\psi = \psi(x_1, x_2, x_3, x_4)$ are nonzero real-valued functions to be determined. Now, we define a basic tensor $g_{\lambda\mu}$ in X_4 by

$$g_{\lambda\mu} = h_{\lambda\mu} + k_{\lambda\mu}, \tag{4.3}$$

where $h_{\lambda\mu}$ is defined by (4.1), and $k_{\lambda\mu}$ is defined by

$$k_{\lambda\mu} = 2A_{[\lambda} B_{\mu]}, \tag{4.4}$$

that is,

$$((k_{\lambda\mu})) = \begin{pmatrix} 0 & 0 & -\phi & \phi \\ 0 & 0 & -\psi & \psi \\ \phi & \psi & 0 & 0 \\ -\phi & -\psi & 0 & 0 \end{pmatrix}, \tag{4.5}$$

which is obviously of the third class :

$$(4.6) \qquad (a)\ Det(k_{\lambda\mu}) = 0, \qquad (b)\ K(= \frac{1}{4}k_{\alpha\beta}k^{\alpha\beta}) = 0.$$

Then all the Christoffel symbols $\{_\lambda{}^\nu{}_\mu\}$ vanish. Hence the components of the first covariant derivatives with respect to $\{_\lambda{}^\nu{}_\mu\}$ are ordinary derivatives, and $H^\omega_{\lambda\mu\nu} = 0$. Furthermore,

$$(4.7) \qquad (a)\ A^\lambda(= h^{\lambda\nu}A_\nu) : (0,0,1,1), \qquad (b)\ B^\lambda(= h^{\lambda\nu}B_\nu) : (\phi,\psi,0,0)$$

$$(4.8) \qquad (a)\ A_\alpha A^\alpha = A_\alpha B^\alpha = 0, \qquad (b)\ k_{\lambda\alpha}A^\alpha = 0, \qquad (c)\ \partial_\lambda A_\mu = 0.$$

The following theorem is immediate consequences of Theorem 3.3 and Theorem 3.4, in virtue of (2.7)(a), (3.8), and (4.3).

Theorem 4.1. *In X_4, for the basic tensor $g_{\lambda\mu}$ given by (4.3), the connection (3.3) is given by*

$$(4.9) \qquad \Gamma^\nu_{\lambda\mu} = 2\delta^\nu_{[\lambda}k_{\mu]\alpha}Y^\alpha + k_{\lambda\mu}Y^\nu,$$

and this connection (4.9) is a solution of (2.6) and (2.7)(a) if and only if the vector Y^ν defining (4.9) satisfies the following conditions

$$(4.10) \qquad (a)\ k_{\mu\alpha}Y^\alpha = 0, \qquad (b)\ \partial_\nu\, k_{\lambda\mu} = -2k_{\nu[\lambda}\, Y_{\mu]}.$$

If these conditions (4.10) are satisfied, then the connection (4.9) is given by

$$(4.11) \qquad \Gamma^\nu_{\lambda\mu} = k_{\lambda\mu}Y^\nu,$$

which is an Einstein's connection with zero torsion vector.

Remark 4.2. In X_4, since the tensor $k_{\lambda\mu} \neq 0$ is skew-symmetric, we know from elementary algebra that the rank of the matrix $((k_{\lambda\mu}))$ can be either four or two. In virtue of (4.6)(a), in our case the rank must be two. Therefore, the homogeneous equations (4.10)(a) have at least two distinct solutions $Y^\nu_1 : (0,0,1,1)$ and $Y^\nu_2 : (\psi,-\phi,0,0)$. Every linear combination

$$(4.12) \qquad Y^\nu = \rho Y^\nu_1 + \eta Y^\nu_2 : (\eta\psi, -\eta\phi, \rho, \rho)$$

with scalars ρ, η is also a solution of (4.10)(a). On the other hand, if (4.12) is a solution of the condition (4.10)(b), then, in virtue of $k_{12} = 0$ and $Y_\lambda = h_{\lambda\nu}Y^\nu : (\eta\psi, -\eta\phi, \rho, -\rho)$, we obtain

$$(4.13) \qquad 0 = \partial_3\, k_{12} = -2k_{3[1}\, Y_{2]} = -\phi(-\eta\phi) + \psi(\eta\psi) = \eta(\phi^2 + \psi^2),$$

which implies that $\eta = 0$. Therefore the solutions of the conditions (4.10) are of the form :

$$Y^\nu = \rho Y_1^\nu = \rho A^\nu, \tag{4.14}$$

for some nonzero real-valued function $\rho = \rho(x_1, x_2, x_3, x_4)$ to be determined.

THEOREM 4.3. *In X_4, the vector $Y^\nu = \rho A^\nu$ given by (4.14) is a solution of the conditions (4.10) if and only if the vector B_λ given by (4.2)(b) satisfies the following condition*

$$\partial_\omega B_\mu = \rho A_\omega B_\mu. \tag{4.15}$$

Proof. Suppose that (4.15) is satisfied. Differentiating both sides of (4.4), and making use of (4.8)(c) and (4.15), we obtain

$$\partial_\omega k_{\lambda\mu} = A_\lambda(\rho A_\omega B_\mu) - A_\mu(\rho A_\omega B_\lambda) = -k_{\omega\lambda}(\rho A_\mu) + k_{\omega\mu}(\rho A_\lambda). \tag{4.16}$$

Hence, in virtue of Remark 4.2 and the relation (4.16) , the vector $Y^\nu = \rho A^\nu$ is a solution of (4.10). Conversely, suppose that the vector $Y^\nu = \rho A^\nu$ is a solution of (4.10). Since $B_\mu = k_{3\mu}$, we obtain

$$\begin{aligned} \partial_\omega B_\mu &= \partial_\omega k_{3\mu} = -k_{\omega 3} Y_\mu + k_{\omega\mu} Y_3 \\ &= -(A_\omega B_3 - A_3 B_\omega)(\rho A_\mu) + (A_\omega B_\mu - A_\mu B_\omega)(\rho A_3) = \rho A_\omega B_\mu, \end{aligned} \tag{4.17}$$

in virtue of (4.4), (4.10)(b), and (4.2). Hence the condition (4.15) is satisfied. □

THEOREM 4.4. *In X_4, the condition (4.15) is satisfied if and only if the functions ρ, ϕ, and ψ, given in (4.15), satisfy the following conditions, respectively,*

$$\partial\phi/\partial x^1 = 0, \quad \partial\phi/\partial x^2 = 0, \quad \partial\phi/\partial x^3 = \rho\phi, \quad \partial\phi/\partial x^4 = -\rho\phi, \tag{4.18}$$

$$\partial\psi/\partial x^1 = 0, \quad \partial\psi/\partial x^2 = 0, \quad \partial\psi/\partial x^3 = \rho\psi, \quad \partial\psi/\partial x^4 = -\rho\psi \tag{4.19}$$

$$\partial\rho/\partial x^1 = 0, \quad \partial\rho/\partial x^2 = 0, \quad \partial\rho/\partial x^3 + \partial\rho/\partial x^4 = 0. \tag{4.20}$$

Proof. In virtue of (4.15), we obtain

$$\partial\phi/\partial x^\omega = \partial_\omega B_1 = \rho A_\omega B_1 = \rho\phi A_\omega, \tag{4.21}$$

which imply (4.18), in virtue of (4.2)(a). Similarly, we obtain (4.19). Next, differentiating both sides of (4.21), we obtain

$$\frac{\partial^2\phi}{\partial x^\nu \partial x^\omega} = \frac{\partial\rho}{\partial x^\nu}\phi A_\omega + \rho^2\phi A_\nu A_\omega, \tag{4.22}$$

in virtue of (4.8)(c) and (4.21), and we also obtain

$$\frac{\partial^2\phi}{\partial x^\omega \partial x^\nu} = \frac{\partial\rho}{\partial x^\omega}\phi A_\nu + \rho^2\phi A_\omega A_\nu. \tag{4.23}$$

Hence we obtain

$$0 = \frac{\partial^2\phi}{\partial x^\nu \partial x^\omega} - \frac{\partial^2\phi}{\partial x^\omega \partial x^\nu} = (\frac{\partial\rho}{\partial x^\nu}A_\omega - \frac{\partial\rho}{\partial x^\omega}A_\nu)\phi, \tag{4.24}$$

which implies that, since $\phi \neq 0$,

$$\frac{\partial\rho}{\partial x^\nu}A_\omega - \frac{\partial\rho}{\partial x^\omega}A_\nu = 0. \tag{4.25}$$

From the above condition (4.25), we obtain that if $\nu = 1$ and $\omega = 3$, then $\partial\rho/\partial x^1 = 0$, if $\nu = 2$ and $\omega = 3$, then $\partial\rho/\partial x^2 = 0$, and if $\nu = 3$ and $\omega = 4$, then $\partial\rho/\partial x^3 + \partial\rho/\partial x^4 = 0$. Hence we obtain (4.20). Obviously, the converse is true. □

THEOREM 4.5. *In X_4, for the basic tensor $g_{\lambda\mu}$ given by (4.3), the connection (4.11) which is a solution of (2.6) and (2.7)(a) is given by*

$$\Gamma^\nu_{\lambda\mu} = 2\rho A_{[\lambda}B_{\mu]}A^\nu, \tag{4.26}$$

where ρ satisfies the condition (4.20). And the curvature tensor $R^\alpha_{\lambda\mu\alpha}$ with respect to this connection (4.26) is given by

$$R^\omega_{\lambda\mu\nu} = 2\{(\partial_{[\mu}\rho)B_{\nu]}A_\lambda - (\partial_{[\mu}\rho)A_{\nu]}B_\lambda\}A^\omega + 2\rho^2 A_{[\mu}B_{\nu]}A_\lambda A^\omega, \tag{4.27}$$

and its contracted curvature tensor $R_{\lambda\mu}$ satisfies

$$R_{\lambda\mu} = 0. \tag{4.28}$$

Proof. Substituting (4.4) and (4.14) into (4.11), we obtain (4.26), in virtue of Remark 4.2, Theorem 4.3, and Theorem 4.4. Substituting (4.26) into (2.8), we obtain (4.27) by a straightforward computation. In the next, Contracting for (4.27) for ω and ν, we obtain

$$R_{\lambda\mu} = -2(\partial_\alpha\rho)A^\alpha A_{[\lambda}B_{\mu]}. \tag{4.29}$$

On the other hand, in virtue of (4.7)(a) and (4.20), we obtain

$$(\partial_\alpha\rho)A^\alpha = \partial\rho/\partial x^3 + \partial\rho/\partial x^4 = 0 \tag{4.30}$$

Hence we obtain (4.28). □

REMARK 4.6. The set of the functions ϕ satisfying (4.18) is not empty. For example, when ρ= constant, the function

$$\phi(x_1, x_2, x_3, x_4) = e^{\rho(x^3 - x^4)} \tag{4.31}$$

satisfies (4.18). Similarly, we can define the function ψ satisfying (4.19).

Conclusion. In virtue of Theorem 4.5, if X_4 is endowed with a non-symmetric tensor $g_{\lambda\mu} = h_{\lambda\mu} + k_{\lambda\mu}$ such that (4.1) and (4.5), where ϕ and ψ satisfy the conditions (4.18) and (4.19), respectively. Then a solution $\Gamma^{\nu}_{\lambda\mu}$ of (2.6) and (2.7)(a) is given by (4.26), where ρ satisfies the condition (4.20). In the next, since the contracted curvature tensor $R_{\lambda\mu}$ with respect to the connection (4.26) satisfies $R_{\lambda\mu} = 0$, the field equation (2.7)(c) is satisfied automatically, and the field equation (2.7)(b) is equivalent to $\partial_{[\lambda}P_{\mu]} = 0$. Since the field equation (2.7)(b) is satisfied by a vector $P_\mu = \partial_\mu P$, the vector $P_\mu = \partial_\mu P$ is an Einstein's vector.

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